A stable $\aleph_0$-categorical pseudo-plane

1. Introduction

In [L] it was conjectured that every stable, $\aleph_0$-categorical structures is $\aleph_0$-stable. It was later shown that

stable: for every 1st-order formula $\varphi(\overline{x},\overline{y})$ with 2n variables, considered as a binary relation on M^n, there is a finite bound $n(\varphi)$ on the size of subsets of M^n linearly ordered by φ.

$\aleph_0$-categorical: for each n, only finitely many formulas, up to equivalence.

$\aleph_0$-categorical, $\aleph_0$-stable structures.
i) There are only countably many.
ii) (Zil'ber) building blocks: infinite-dimensional vector space V over a finite field F.
appears as $[V]^d$, $d \geq 1$.
How do the building blocks occur inside the given model M?
iii) (Cherlin-Harrington-Lachlan) Equivalence relations.

Discuss (iii). How else might the "blocks" occur inside a model?
Equivalence relations: 1 point of the block determines the block.
generalization:
A set S; a set B of subsets of S, the "blocks"; and any 2 points of a block determine the block (but one doesn't).
Equivalently: any two blocks intersect in at most one point.
Example: S = plane, B = set of lines
pseudo-planes: any two blocks intersect in finitely many points, while any block is infinite, and every point lies in infinitely many blocks.

So (iii) = there are no $\aleph_0$-categorical, $\aleph_0$-stable pseudo-planes.

((ii),(iii) are closely related.)
Assuming stability alone, a priori there are no irreducible building blocks. Still, Lachlan proved that every stable $\aleph_0$-categorical structure without pseudoplanes is $\aleph_0$-stable.

construction of a stable, $\aleph_0$-categorical pseudo-plane.

2. The construction.

We will actually build a graph Γ rather than a pseudoplane. (To settle conventions, a graph will be a set A together with a symmetric, irreflexive binary relation R on A. A subgraph will be a subset B of A with the relation $R \cap B^2$.) Given $\Gamma=(\Gamma,R)$, we define the set of <u>blocks</u> to be the subsets of Γ of the form $\{y: aRy\}$, with $a \in \Gamma$. The (strong) pseudo-plane condition translates to:

i) There are no squares, i.e. no distinct elements a_0,a_1,a_2,a_3 such that a_iRa_{i+1} (with + mod 4)

ii) For all a, there are infinitely many b with aRb.

To get Γ, we will define a class $\underline{c}$ of finite graphs; and a relation $A \le B$ between elements A,B of $\underline{c}$ with A a subgraph of B. (In the end, c will be the class of finite subgraphs of Γ, and $A \le B$ will mean: when B is embedded in Γ generically, A is algebraically closed in B.) M will then be built as a "direct limit" of $(\underline{c},\le)$ (($\underline{c},\le$) will not quite be a directed system, but the amalgamation property (C1) below is an adequate substitute.)

Required properties.

C1) $\underline{c}$ will be closed under substructures and isomorphisms, and will have the amalgamation property: whenever $A \le B_1$, $A \le B_2$, and $B_1,B_2 \in \underline{c}$, then there exists $E \in \underline{c}$ and embeddings $f_i:B_i \to E$ such that $fB_i \le E$ and $f_1|A = f_2|A$.

C2) Given $A \subseteq B \in \underline{C}$, can find $B' \leq B$, $A \subseteq B'$, with $|B'|$ bounded as a function of $|A|$.

C3) There is no "square" in $\underline{C}$, i.e. no structure with universe $\{1,2,3,4\}$ such that $\{1,2\},\{2,3\},\{3,4\},\{1,4\}$ are among the edges.

These will give at least an $\aleph_0$-categorical graph Γ. Below we will give a simple proof that the structure obtained is in fact stable. The deeper reason (and the motivation for the construction) is that M admits a rank function on definable sets, with all the properties of Morley rank in finite rank (including additivity), but real valued. The real number α used as a parameter in the construction will turn out to be the co-rank of $\{(x,y): xRy\}$ in Γ^2.

Let α be a real number in the range $(1/2,2/3)$ satisfying:

(*) There are infinitely many rationals p/q $(q>0)$ with $0<\alpha-p/q<e^{-q}$.

Let $\psi(x)$ be the least rational number p/q with $p/q>\alpha$ and $1 \leq q \leq x$. Let $\Psi(x) = \int_{[1,x]}(1-\alpha/\psi(t))dt$. (For $x \leq 1$, $\Psi(x)=0$.) Let Λ be the lattice generated by $(1,1)$ and $(0,\alpha)$; $(x,y) \in \Lambda$ iff $x \in Z$ and $y-x \in \alpha Z$. Let $J=\{(x,y) \in \Lambda: x \geq 0, y \geq \Psi(x)\}$.

$\psi(1)=1$. $\psi(2)=1$. $\psi(3)=2/3$. $\psi(4)=2/3$. $\psi(5)=2/3$ or $3/5$.

$\Psi(1)=1-\alpha$

$\Psi(2)=2-2\alpha$.

$\Psi(4)=2-2\alpha+3(2/3-\alpha) = 4-5\alpha$.

$\Psi(8) \leq 4-5\alpha+2(2-3\alpha) = 8-11\alpha < 8-12\alpha$

Ok; this will have squares; but the intersection of two lines will have ≤ 5 points.

<u>Lemma 1</u> Let $a,b,c \in J$ and suppose both $b-a$ and $c-a$ are in the open upper right quadrant. Then $b-a+c \in J$.

<u>Proof</u> Let u_1,u_2 denote the x,y-coordinates of u, respectively. We may assume $c_1 \geq b_1$ and $c_1 \geq 1$ (the case $c_1=b_1=0$ being trivial.) Let $b-a=p(1,1)-q(0,\alpha)$. Then $p-q\alpha>0$ and $p>0$; and $p=b_1 \leq c_1$; so $\varphi(c_1) \leq p/q$. Let s be the slope of b-a. Then $s = (p-q\alpha)/p = 1-$

$(q/p)\alpha \geq (1-\alpha/\psi(c_1))$. So $\Psi(x)=\Psi(c_1)+\int[c_1,x](1-\alpha/\psi(x)) \leq \Psi(c_1)+\int[c_1,x](1-\alpha/\psi(c_1)) \leq$

$\leq \Psi(c_1)+\int[c_1,x]s = \Psi(c_1)+(x-c_1)s = \Psi(c_1)+(b-a)_2 \leq c_2+(b-a)_2 = (b-a+c)_2$

and $b-a+c \in J$.

Lemma 2 Ψ is unbounded.

Proof Let q be such that $0<\alpha-p/q<e^{-q}$, and q large. Then I claim that $\Psi(e^q/q)\geq\Psi(q)+1/4$. If $q\leq x\leq e^q/2q$, let $\psi(x)=p'/q'$. Then $\psi(x)-\alpha = (\psi(x)-p/q)-(\alpha-p/q) > 1/(qq')-(e^{-q})\geq 1/(qx)-1/(2qx)= 1/(2qx)$. So $1-\alpha/\psi(x)=(\psi(x)-\alpha)/\psi(x)\geq(\psi(x)-\alpha)\geq 1/(2qx)$, and

$\Psi(e^q/2q)=\Psi(q)+\int_{[q,e^q/2q]}(1-\alpha/\psi(x)) \geq \Psi(q)+\int_{[q,e^q/2q]} 1/(2qx) = \Psi(q)+(1/2q)[\log(e^q/2q)-\log(q))] \geq \Psi(q)+1/4$. Since there are infinitely many such q's, Ψ is unbounded.

Compute some values of Ψ

$\psi(1)=1$. $\psi(2)=1$. $\psi(3)=2/3$. $\psi(4)=2/3$. $\psi(5)=2/3$ or $3/5$.

$\Psi(1)=1-\alpha$

$\Psi(2)=2-2\alpha$.

$\Psi(4)=2-2\alpha+3(2/3-\alpha) = 4-5\alpha$.

$\Psi(8)\leq 4-5\alpha+2(2-3\alpha) = 8-11\alpha < 8-12\alpha$

Ok; this will have squares; but the intersection of two lines will have ≤ 5 points.

We can now define $\underline{C}$ and $\leq$. For a finite graph A, let x(A) be the number of points of A, e(A) the number of edges, $y(A)=x(A)-\alpha\cdot e(A)$, and $v(A)=(x(A),y(A))$. Let $\underline{C}$ be the class of graphs A such that for every subgraph A' of A, $v(A')\in J$, . If A is a finite subgraph of a graph B, write $A\leq B$ if for every finite $B'\subseteq B$ with $A\subset B'\subseteq B$, $y(A)<y(B')$. If A,A' are finite subsets of B, let $y(A'/A)=y(A'\cup A)-y(A)$.

(C1) Closure under substructures is clear.

If A is a subgraph of both B_1 and B_2, the free amalgam of B_1,B_2 over A is the graph E whose universe is the disjoint union $B_1\dot{\cup}B_2$, and whose edges are just those of B_1 and those of B_2.

Lemma 5

i) If $A\leq B$ and $X\subseteq B$ then $(X\cap A)\leq X$. More generally,

ii) $y(Z/A)\geq y(Z/A')$ if $A\subseteq A'\subseteq B$, A' finite, and $Z\cap A'\subseteq A$.

ii) If A,B are finite, $A\leq B\leq B'$, then $A\leq B'$.

Lemma 6 Let $A,B_1,B_2\in\underline{C}$, $A\leq B_1, A\leq B_2$. Then $E\in\underline{C}$, and $B_1\leq E$.

Proof Let X be a substructure of E, $X_1=X\cap B_1$, $X_0=X\cap A$. There are no edges between points of X_1-X_0 and X_2-X_0, so one computes easily: $v(X)=v(X_1)-v(X_0)+v(X_2)$. By 5(i), $X_0\leq X_1$ and $X_0\leq X_2$; this means that $y(X_1)\geq y(X_0)$. Certainly $x(X_1)\geq x(X_0)$. So $0<\theta(v(X_0),v(X_1))\leq\pi/2$. The fact that X_1 has at least as many edges as X_0 translates to $\theta(v(X_0),v(X_1))\leq\pi/4$. By lemma 1, $v(X)\in H$.

The previous meaning of M is no longer needed.

Lemma 7 There exists a countable graph M such that:
i) Every finite subgraph of M is in $\underline{c}$.
ii) Let $A\leq M$, A finite, and $A\leq B$, $B\in\underline{c}$. Then there exists $B'\leq M$ such that B and B' are isomorphic over A.

Proof Standard.

Lemma 8 M is uniquely described by 7(i),(ii). If $A\leq M$, $A'\leq M$, and A,A' are finite, then any graph isomorphism between A and A' extends to an automorphism of M. M is $\aleph_0$-categorical.

Proof The last statement follows from the first and the definition of $\aleph_0$-categoricity, as (i) and (ii) are properties of the first-order theory of M. The first and second sentences follow from a standard back-and-forth argument, building an isomorphism between M and M' using approximations $f:B\rightarrow B'$ with $B\leq M$, $B'\leq M'$. For this to work we must show:
*) Let $A\subseteq M$ be finite. Then there exists a finite $B\subseteq M$, $A\subseteq B$, with $B\leq M$.
By lemma 4, there are only finitely many possible values of $y(B)$ for $A\subseteq B\subseteq M$ and $y(B)\leq y(A)$. Pick $B\subseteq M$, $A\subseteq B$, B finite, with $y(B)$ least possible. Then clearly $B\leq M$.

Given a finite $A\subseteq M$, define $d(A)=\inf\{y(B):\ A\subseteq B\subseteq M,\ B\text{ finite}\}$.

<u>Lemma 9</u> For every finite $A \subseteq M$ there exists a unique smallest finite $B \supseteq A$ with $y(B)=d(A)$.

By the argument in lemma 8, the infimum in the definition of $d(A)$ is in attained. Suppose B_1,B_2 are two candidates: $y(B_1)=y(B_2)=d(A)$, $B_i \supseteq A$, and B_i has no proper subsets with these properties. Let $A'=B_1 \cap B_2$. If $A'=B_1=B_2$ we are done. If not say $A' \neq B_1$. Then $y(B_1/A')<0$. (Otherwise $y(B_1) \geq y(A')$.) By lemma 5(ii), $y(B_1/B_2)<0$. So $y(B_1 \cup B_2)<y(B_2)=d(A)$, a contradiction.

Write $cl(A)$ for this unique B. So $cl(A) \subseteq acl(A)$. (The converse is also true but will not be needed.) Note that $cl(cl(A))=cl(A)$, and if $A \subseteq B$ then $cl(A) \subseteq cl(B)$ (using 5(ii)). For infinite B, let $cl(B)=\cup\{cl(A): A \subseteq B, A \text{ finite}\}$. Call a set B closed if $cl(B)=B$. Also write $\overline{B}$ for $cl(B)$.

Write $d(A/B)$ for $d(A \cup B)-d(B)$. In view of 10(b), define $d(A/B)$ for possibly infinite B to be $\inf\{d(A/B'): B' \text{ finite}, B' \subseteq B\}$. For finite A_1,A_2 write $A_1 \perp A_2 | B$ for: $d(A_1/A_2B)=d(A_1/B)$, and $cl(A_1 \cup B) \cap cl(A_2 \cup B) \subseteq cl(B)$. In general, let $A_1 \perp A_2 | B$ iff $A_1' \perp A_2' | B$ for every finite $A_1' \subseteq A_1$ and $A_2' \subseteq A_2$.

<u>Lemma 10</u> Let A be finite.

a) $d(A/B) \geq 0$ always.

b) If $B \subseteq B'$ then $d(A/B) \geq d(A/B')$.

c) $d(A/B)=d(A/cl(B))=d(cl(A)/B)=d(A \cup B/B)$. If A_1,A_2 are finite, then $d(A_1A_2/B)=d(A_2/B)+d(A_1/A_2 \cup B)$.

d) Suppose C,B_1,B_2 are closed, $C \subseteq B_1,B_2$, and $B_1 \perp B_2 | C$. Then $B_1 \cup B_2$ is closed.

e) Suppose $A_1 \perp A_2 | B$, B is closed, and $tp(A_1/B)$ and $tp(A_2/B)$ are known. Then there is only one possibility for $tp(A_1A_2/B)$.

<u>Lemma 12</u> M is a pseudoplane.

The points and the lines are both interpreted as the elements of M, and incidence is the graph adjacence relation. To show that two lines meet in at most one point (or dually) is to show that the graph embeds no squares. Indeed if A is a square then $x(A)=4$, $e(A)\geq 4$, $y(A)\leq 4-4\alpha\leq 2$. But one checks easily that the point (4,2) is in K. To show that through every point there are infinitely many lines, let $a\in M$, and let A be the substructure of M with universe {a}. Then $A\leq M$, because the only point (x,y) in H with $y\leq 1$ and $x\geq 1$ is the point (1,1). Let $B=\{a,b_1,\ldots,b_N\}$, where $b\notin M$, and a is adjacent to each b_i, but there are no other edges. Then $v(B)=(N+1,(N+1)-N\cdot\alpha)\in H$, and $A\leq B$. So B embeds into M over A.

Remarks

a) If α is rational, then α satisfies our hypothesis (*), and we get an $\aleph_0$-categorical pseudo-plane; but it is unstable.

b) It can be shown that if the pseudoplane M_α given by this construction with parameter α is $\aleph_0$-categorical, then for every $\epsilon>0$ the following approximation to (*) holds:

(*') For every $\epsilon>0$ there are infinitely many rationals p/q with $\alpha>p/q$ and $q<\exp(q^{1-\epsilon})$.

and $y(4)>2$. I.e. not 4 edges.

Can this be done?

a,b,c.

How do you get a 4-element substructure?

Look: how would you get a square by free amalgamation? clearly only over 2,3,3, etc. but this does not work. Because of >, and 1/2.

Then $\{.\}\leq M$, because the only point (x,y) in H with $y\leq 1$ and $x\geq 1$ is the point (1,1).

I.e.: $\Psi(x)>1$ for $x\geq 2$. clear. except for $x=2$. But then we could have at most one edge, so $y\geq 1.5$ anyway.

Stability

Let A_i ($i\in Z$) be finite sets (implicitly enumerated in some way.) Suppose $tp(A_i,A_j)=q$ for all $i<j$. We will show that $tp(A_i,A_j)=q$ for some $i>j$.

Let $A_{i,j}=acl(A_i\cup A_j)$, and let $A_{i,j}=(a_{i,j,k}: k<K)$. By Ramsey's theorem we may that for $i<j<j'<j''$,

$A_{i,j}\cap A_{i,j'}=A_{i,j'}\cap A_{i,j''}$

(the alternative is that $A_{i,j}\cap A_{i,j'}\neq A_{i,j'}\cap A_{i,j''}$ for all $i<j<j'<j''$; this means that $A_{i,j}\cap A_{i,j'}$ gives infinitely many distinct subsets of $A_{i,j}$ as $A_{i,j'}$ varies.)

So in fact $A_{i,j}\cap A_{i,j'}$ does not depend on j,j' if $i<j<j'$. Call it $\overline{A}_i$.

Similarly WMA $A_{i,j}\cap A_{i',j}=\tilde{A}_j$ if $i<i'<j$.

Finally, let $\overline{\overline{A}}_i = (\overline{A}_i\cup\tilde{A}_i)$; then WMA $\overline{\overline{A}}_i\cap\overline{\overline{A}}_j=C_0$ if $i\neq j$.

Similarly for $j<i<j'$.

Claim $A_{i,j} \subseteq c(A_i)\cup c(A_j)$

Proof Ow, rank goes down quadratically.

So now we are dealing with the atomic type of $C_i\cup C_j$.

Use Δ-lemma. Also, there can be no edges between C_i, C_j (after another Ramsey.) And the type of C_i may be assumed constant.

Fix such n, a and b. Let $a=a_1,\dots,a_{n-1}$ begin a Morley sequence in stp(a/b).

Claim 1. $b \notin acl(a_1,..,a_{n-1})$.

Proof. Otherwise 2n-1 = n+n-1 =rank(b) + rank($a_1 \dots a_{n-1}$/b) = rank($ba_1 \dots a_{n-1}$) = rank($a_1 \dots a_{n-1}$) $\leq$ 2(n-1) = 2n-2, a contradiction.

Claim 2. Let stp($b'/a_1 \dots a_{n-1}$) = stp($b/a_1 \dots a_{n-1}$) with b and b' independent over $a_1 \dots a_{n-1}$. Then $a_1 \dots a_{n-1}$ is contained in acl(b,b').

Proof. Note $b' \notin acl(b)$. Fix i. Let b_1 = Cb(stp(a_i/b), and b_1' = Cb(stp(a_i/b'). Then b_1 is interalgebraic with b, and b_1' is interalgebraic with b'. Thus $b_1' \notin acl(b_1)$. Thus stp(a_i/b) and stp(a_i/b') have no common nonforking extension. Namely $a_i \in acl(b,b')$.

Claim 3. If σ is any permutation of $\{a_1,\dots,a_{n-1}\}$ then σ is {b,b'}-elementary.

Proof. Let X = $(a_1,..,a_{n-1})$. As $a_1,\dots,a_{n-1}$ is a Morley sequence over b, (*) tp(σ(X),b) = tp(X,b) and tp(σ(X),b') = tp(X,b'). Let q(y) = tp(b/X), and let σ(q) be the image of q under the map σ. So b,b' are independent realisations of q with the same strong type over X. b,b' are also independent realisations of σ(q) with the same strong type over σ(X). By (*) clearly tp(X,b,b') = tp(σ(X),b,b'), proving the claim.

This gives Proposition 3.1 (b). Proceed as before with proof of Proposition A.

p.8, References. Add the full Baldwin reference.